Economics 202a, Fall 1998

Lecture 2: Solow Model Handout

Basics:

$$Y_t = F(K_t, A_t L_t)$$

$$\frac{dA_t}{dt} = gA_t$$

$$\frac{dL_t}{dt} = nL_t$$

The capital-accumulation equation is more interesting:

$$\frac{dK_t}{dt} = sY_t - \delta K_t$$

Now we are in the simplification business, so we need to simplify: we want a system with only one variable.

So we assume constant returns to scale, so that:

$$\frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t} F(K_t, A_t L_t) = F\left(\frac{K_t}{A_t L_t}, 1\right)$$

and we adopt so-called "intensive" notation with lower-case letters for values of variables per unit of "effective labor":

$$\frac{Y_t}{A_t L_t} = y_t = F\left(\frac{K_t}{A_t L_t}, 1\right) = F(k_t, 1) = f(k_t)$$

We can use the identities:

$$\frac{d}{dt}\ln(x) = \frac{1}{x}\frac{dx}{dt}, \ln(xy) = \ln(x) + \ln(y)$$

to calculate the accumulation equation for k = K/(AL):

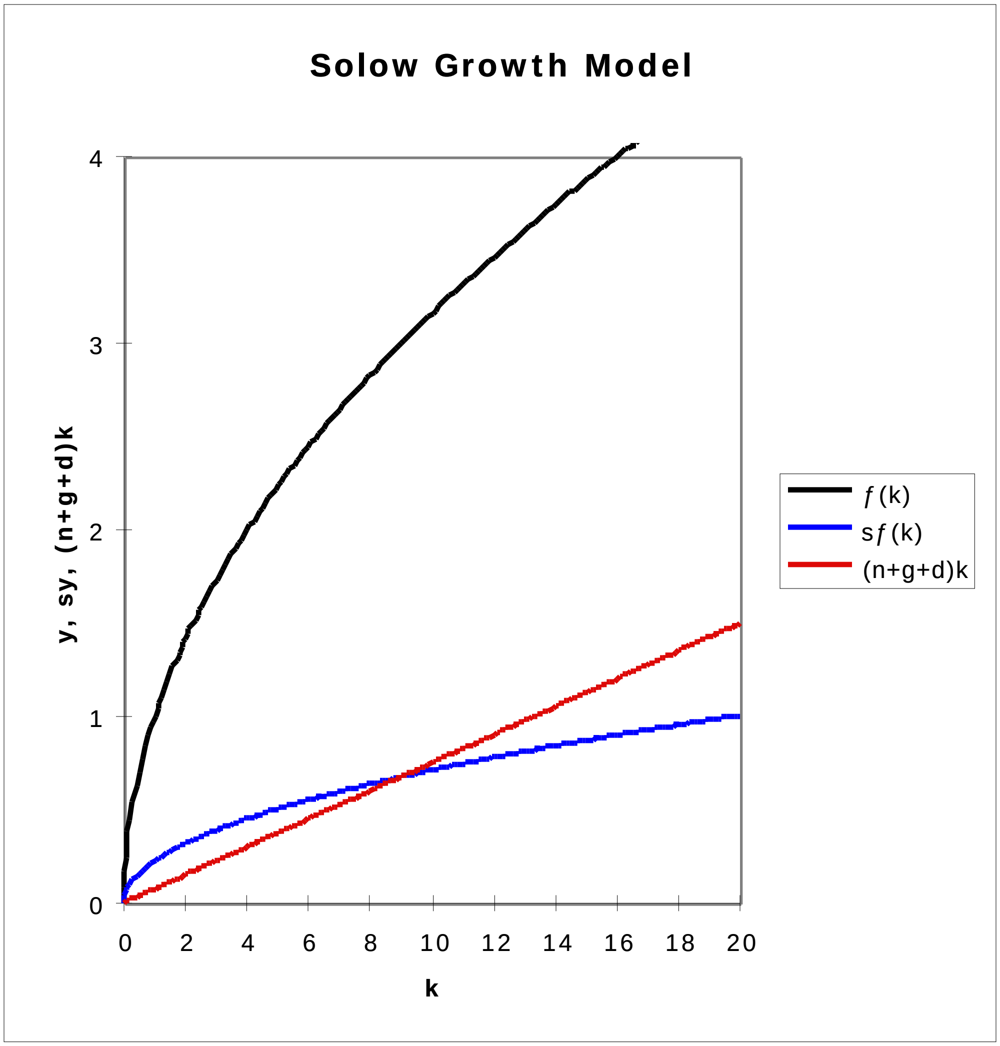
$$\frac{d}{dt}\ln(k_t) = \frac{d}{dt}\ln(K_t) - \frac{d}{dt}\ln(A_t) - \frac{d}{dt}\ln(L_t)$$

$$\frac{1}{k_t}\frac{dk_t}{dt} = \left(s\frac{Y_t}{K_t} - \delta\right) - g - n$$

$$\frac{1}{k_t}\frac{dk_t}{dt} = s\frac{y_t}{k_t} - \delta - g - n$$

$$\frac{dk_t}{dt} = sf(k_t) - (n + g + \delta)k$$

And now we can draw pretty diagrams...



And we can also draw some ugly diagrams. But we want to rule those out. So let's require:

$$f'(0) = \infty$$

$$f'(\infty) = 0$$

$$f'(x) > 0, \quad x \in (0,\infty)$$

$$f''(x) < 0, \quad x \in (0,\infty)$$

And so now we can only draw pretty diagrams.

And we can also note that there is a k* such that:

$$(k_t < k^*) \qquad \left(\frac{dk_t}{dt} > 0\right)$$

$$(k_t > k^*) \qquad \left(\frac{dk_t}{dt} < 0\right)$$

$$(k_t = k^*) \qquad \left(\frac{dk_t}{dt} = 0\right)$$

Steady-state growth path:

$$y^* = f(k^*)$$
$$Y^*_t = A_t L_t y^*$$
$$K^*_t = A_t L_t k^*$$
$$\frac{K^*_t}{Y^*_t} = z^* = \frac{k^*}{y^*}$$
$$\frac{Y^*_t}{L_t} = A_t y^*$$
$$\frac{K^*_t}{L_t} = A_t k^*$$

Cobb-Douglas Case:

We can actually get considerably further if we want to restrict ourselves to a smaller calss of production functions. Say, the Cobb-Douglas case:

$$F(K_t, A_t L_t) = (K_t)^{\alpha} (A_t L_t)^{1-\alpha}$$

Then the "intensive" form is particularly simple:

$$\frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t}\left(K_t^{\alpha}(A_t L_t)^{1-\alpha}\right) = \left(\frac{K_t}{A_t L_t}\right)^{\alpha}\left(\frac{A_t L_t}{A_t L_t}\right)^{1-\alpha}$$
$$y_t = (k_t)^{\alpha}$$

Then we know that:

$$\frac{d}{dt}\ln(k_t) = \frac{s f(k_t)}{k_t} - (n + g + \delta)$$
$$\frac{d}{dt}\ln(k_t) = s k_t^{\alpha-1} - (n + g + \delta)$$
$$\frac{d}{dt}\ln(y_t) = \frac{d}{dt}\ln(k_t^{\alpha}) = \frac{d}{dt}\alpha(\ln(k_t)) = \alpha\frac{d}{dt}\ln(k_t)$$

So let's write z = k/y = K/Y for the capital-ouput ratio, and see that:

$$\frac{d}{dt}\ln(z_t) = \frac{d}{dt}\ln(k_t) - \frac{d}{dt}\ln(y_t) = (1-\alpha)\frac{d}{dt}\ln(k_t)$$

$$\frac{d}{dt}\ln(z_t) = (1-\alpha)\left(s k_t^{\alpha-1} - (n + g + \delta)\right)$$
$$\frac{d}{dt}\ln(z_t) = (1-\alpha)\left(s z_t^{-1} - (n + g + \delta)\right)$$
$$\frac{dz_t}{dt} = (1-\alpha)(s - (n + g + \delta)z_t)$$

And this is our goal: we can write:

$$z_t = \frac{s}{n+g+\delta} + \varepsilon_t$$

$$\frac{d\varepsilon_t}{dt} = (\alpha - 1)(n+g+\delta)\varepsilon_t$$

$$\varepsilon_t = A_0 d^{(\alpha-1)(n+g+\delta)t}$$

$$z_t = \frac{s}{n+g+\delta} + A_0 e^{(\alpha-1)(n+g+\delta)t}$$

$$\frac{K_t}{Y_t} = \frac{k_t}{y_t} = z_t = \frac{s}{n+g+\delta} + \left[\frac{K_0}{Y_0} - \frac{s}{n+g+\delta}\right] e^{(\alpha-1)(n+g+\delta)t}$$

And with this we can calculate, especially once we realize that:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha} = \left(\frac{K_t}{Y_t} Y_t\right)^{\alpha} (A_t L_t)^{1-\alpha}$$

$$Y_t = Y_t^{\alpha}\left(z_t\right)^{a}(A_t L_t)^{1-\alpha}$$

$$Y_t = A_t L_t \left(z_t\right)^{\frac{\alpha}{1-\alpha}}$$

$$\frac{Y_t}{L_t} = A_t \left(z_t\right)^{\frac{\alpha}{1-\alpha}}$$

$$y_t = \left(z_t\right)^{\frac{\alpha}{1-\alpha}}$$